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An extended geometric criterion for chaos in the Dicke model

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Abstract
We extend HBLSL's (Horwitz, Ben Zion, Lewkowicz, Schiffer and Levitan) new Riemannian geometric criterion for chaotic motion to Hamiltonian systems of weak coupling of potential and momenta by defining the 'mean unstable ratio'. We discuss the Dicke model of an unstable Hamiltonian system in detail and show that our results are in good agreement with that of the computation of Lyapunov characteristic exponents.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In recent decades much attention has been paid to the investigation of Hamiltonian chaos based on the Riemannian geometric approach. A differential geometric approach has been proposed and successfully applied to the study of Hamiltonian systems [1–4]. This method combines analysis theory with numerical simulations, resulting in a very powerful approach that provides an explanation for the origin of Hamiltonian chaos and an effective method to quantify it. This approach regards the trajectory of a dynamical system as a geodesic on the Riemannian manifold endowed with a suitable metric [5, 6]. The instability properties of geodesics are related to the curvature properties of the underlying manifold through the Jacobi–Lévi-Cività (JLC) equation for the evolution of geodesic separation, and the approximate version of the JLC equation has been used for the systems of large number of degrees of freedom.

In fact, there are other criteria for chaotic motion in Hamiltonian systems. One is the technique of the surface of section (Poincaré plots), which is applied to detect chaotic behavior numerically as a very useful tool for low-dimensional Hamiltonian systems. Another technique is the well-known Lyapunov characteristic exponent. For a Hamiltonian system with N degrees of freedom, the characteristic of the motion is determined by the Maximal

Lyapunov characteristic exponent (MLCE). The motion is regarded as being chaotic if the MLCE is positive [9, 10].

A new Riemannian geometric criterion for chaotic motion in Hamiltonian systems has recently been developed by Horwitz, Ben Zion, Lewkowicz, Schiffer and Levitan (HBLSL) [7], based on the idea that the orbits are determined as geodesics on a dynamically induced surface. HBLSL's method contains the same information of chaotic behavior as the MLCE and provides the unstable ratio [7, 8] to detect the transition from order to chaos in Hamiltonian systems of the form $H = \frac{p^2}{2M} + V(x)$. The new criterion of Hamiltonian dynamics has presented a new insight into the structure of the unstable and chaotic behavior of Hamiltonian dynamical systems.

HBLSL's new geometrical criterion is directly effective for Hamiltonians which are dominated by oscillator-like potentials [8] at small distances, and only effective for Hamiltonians in which the potential is not coupled to momenta. Motivated by the effective analyses of the chaos in Hamiltonian systems, in which the potential is coupled to momenta, we will try to extend HBLSL's new criterion to this case in this paper.

2. HBLSL's theory

For any Hamiltonian of the form $H = \frac{p^2}{2M} + V(x)$, where V is a function of space variables x only, HBLSL consider the geodesic deviation $\xi^l = x'^l - x^l$ between two nearby orbits (correlated by the time parameter t) with the special form of the conformal metric [7] on a given energy surface E. HBLSL have proved that the second-order geodesic deviation equations [7] can be written as

$$\frac{D_M^2 \xi}{D_M t^2} = -\mathcal{V}\mathcal{P}\xi, \tag{1}$$

where the matrix $\mathcal{V}$ is given by

$$\mathcal{V}_{ij} = \left\{ \frac{3}{M^2 v^2} \frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} + \frac{1}{M} \frac{\partial^2 V}{\partial x^i \partial x^j} \right\} \tag{2}$$

and

$$\mathcal{P}^{ij} = \delta^{ij} - \frac{v^i v^j}{v^2} \tag{3}$$

with $v^i \equiv \dot{x}^i$. If at least one of the eigenvalues of the matrix $\mathcal{V}$ is negative, the motion will generally be unstable. Ben Zion and Horwitz define [8]

$$\rho = \frac{\text{volume of region of negative eigenvalue}}{\text{volume of physically accessible region}}. \tag{4}$$

We can call ρ an unstable ratio, and any nonzero ρ indicates instability. Ben Zion and Horwitz have compared the results of the unstable ratio with that of the computation regarding the surface of section for a family of models, and their results have shown that the criterion is much simpler and more efficient than the well-known Lyapunov characteristic exponents for instability [8].

3. The extended theory

For the Hamiltonian, HBLSL applied their criterion to $H = \frac{p^2}{2M} + V(x)$; it is obvious that the potential and the momenta are not coupled. In fact, there is a much wider range of Hamiltonian in which the potential and the momenta are coupled as the form $H = \frac{1}{2}p^2 + V(x, p)$. The new geometric criterion appears to be inapplicable if the potential depends on momenta as well as coordinates. In this paper we extend the criterion to the case in which the potential is weakly coupled to the momenta, a condition expressed by inequality (6).

We consider the Hamiltonian $H = \frac{1}{2}p^2 + V(x, p)$ with $M = 1$, where $V(x, p)$ is the momenta-dependent potential. The differential equations obtained from the above Hamiltonian are

$$\begin{cases} \dot{x}^i = p_i\left(1 + \frac{\partial V}{\partial p_i}\Big/ p_i\right) \\ \dot{p}_i = -\frac{\partial V}{\partial x^i} \end{cases} \qquad (i = 1, \ldots, n). \tag{5}$$

If

$$\frac{\partial V}{\partial p_i}\Big/ p_i \ll 1 \qquad (i = 1, \ldots, n), \tag{6}$$

then

$$1 + \frac{\partial V}{\partial p_i}\Big/ p_i \approx 1 \qquad (i = 1, \ldots, n). \tag{7}$$

In other words, if $\frac{\partial V}{\partial p_i}\big/ p_i$ is much less than 1, then $\frac{\partial V}{\partial p_i}\big/ p_i$ can be neglected. We will obtain an approximation of equation (5):

$$\begin{cases} \dot{x}^i = p_i \\ \dot{p}_i = -\frac{\partial V}{\partial x^i} \end{cases} \qquad (i = 1, \ldots, n). \tag{8}$$

Condition (6) could be named the condition of weak coupling of potential and momenta. In this condition, all the properties of the motion are approximately dependent on $-\frac{\partial V}{\partial x^i}$ $(i = 1, \ldots, n)$ which is a function of x and p. We regard p in the momenta-dependent potential V as a parameter. To distinguish p in the momenta-dependent potential $V(x, p)$ from p in the kinetic energy $\frac{1}{2}p^2$, we use $\tilde{p}$ to denote p in $V(x, p)$. Obviously, for a given energy E, ρ is a function of $\tilde{p}$. We define the mean unstable ratio (MUR) as

$$\bar{\rho} = \frac{\int_{\text{physically}} \rho(\tilde{p})\, \mathrm{d}\tilde{p}}{\int_{\text{physically}} \mathrm{d}\tilde{p}} \tag{9}$$

for the given energy E, where 'physically' means the volume of a physically accessible region. Then we can regard the nonzero MUR as the signature of instability.

4. Numerical results and discussion

In order to test the effectiveness of the MUR method, we compare our results with that of the computation of the surface of section and the MLCE for the Hamiltonian of the Dicke model which satisfies the condition of weak coupling of potential and momenta. The Dicke model describes a collection of N two-level atoms interacting with a single-mode light field, and the classical Hamiltonian [11] of the model is

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}(x^2 + y^2) + 2\lambda xy\sqrt{1 - \frac{y^2 + p_y^2 - 1}{2N}} \tag{10}$$

with the field frequency $\omega = 1$ and the atomic levels splitting $\omega_0 = 1$, where λ denotes the coupling strength between the atom and the field, and N is the number of atoms. Obviously, we have $\frac{\partial V}{\partial p_x}\big/ p_x = 0$ and $\frac{\partial V}{\partial p_y}\big/ p_y = -\frac{\lambda}{N}\frac{xy}{\sqrt{1-\frac{y^2+p_y^2-1}{2N}}}$. When N is larger than λ, and the physically accessible region of the system is located in a small region around zero, condition (6) is satisfied. So we can compute the MUR of the system.

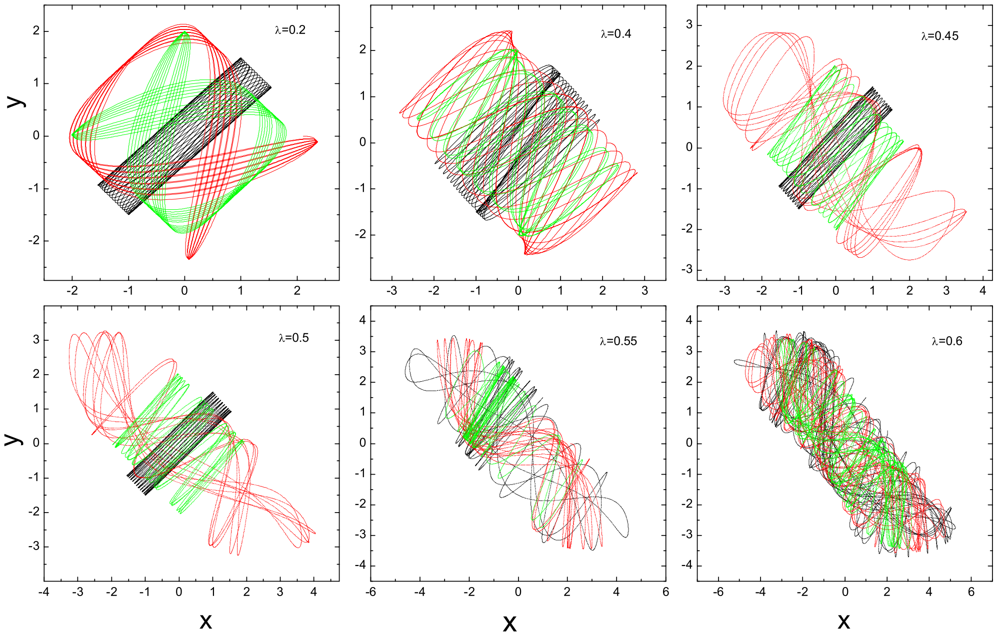

Figure 1. The surface of section for the Dicke model for a sequence of increasing couplings with $N = 100$ and $E = 3.0$.

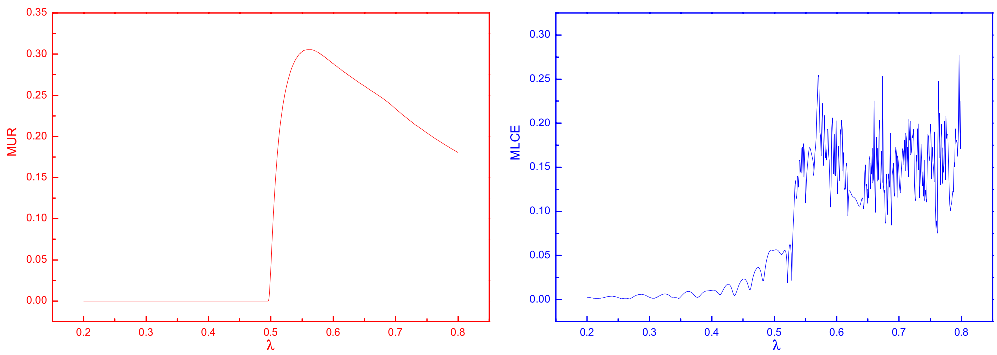

Figure 2. The red line shows the MUR $\bar{\rho}$ plotted as a function of λ, and the blue line shows the MLCE plotted as a function of λ with $N = 100$. The intercept is at $\lambda = 0.5$.

The surface of section with $N = 100$ and $E = 3$ for several coupling strengths λ is shown in figure 1. At small λ ($\lambda = 0.2, 0.4$ in figure 1), the surface of section shows regular, periodic orbits. As the coupling strength λ approaches the critical point $\lambda_c = 0.5$ [11] ($\lambda = 0.45, 0.5$ in figure 1), we can see the emergence of chaotic trajectories. The whole phase space becomes chaotic for the coupling strength $\lambda > \lambda_c$ ($\lambda = 0.55, 0.6$ in figure 1).

We calculate the MUR with the energy $E = 3.0$ while the coupling strength λ is varied from 0.2 to 0.8. The red line shows the MUR $\bar{\rho}$ versus λ in figure 2. The intercept is at $\lambda_c = 0.5$. We also calculate the MLCE by taking the initial values to be the same for all λ: $p_x = 0.5$, $x = 1.0$, $y = 1.0$, the value of p_y is fixed by the energy $E = 3$ and the coupling strength λ. In figure 2 the blue line shows the MLCE versus λ for the same initial values for 5×10^4 time steps.

As shown in figure 2, the MUR vanishes when $\lambda < \lambda_c$ and increases abruptly with λ when $\lambda > \lambda_c$; the MLCE is very small when $\lambda < \lambda_c$; as the coupling strength λ approaches $\lambda_c = 0.5$, the MLCE increases with λ; and the MLCE is large when $\lambda > \lambda_c$. It is apparent that the MUR and MLCE give results in agreement with the numerical technique of the surface of section. The MUR gives a more clear intercept and can be used to detect the transition from order to chaos in the Dicke model as the MLCE. So it is more efficient than the MLCE for predicting the chaotic behavior of the Dicke model.

5. Conclusion

Based on HBLSL's theory for detecting the instability of the Hamiltonian systems that potential is a function of space variables only, we have presented a method which enables the detection of instability through the MUR for the systems of weak coupling of potential and momenta. The MUR is applicable for the Dicke model and gives results in agreement with the numerical technique of the surface of section as the MLCE, and it shows a clearer picture than the MLCE around the critical point, so it can be used as a more efficient criterion for instability. The MUR can be applied to a much wider case in which the potential is weakly coupled to momenta.

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